Nonlinear Resonant Roll Motion of Magnetically Oriented Satellites

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A systematic analysis of the steady-state roll motion of satellites with passive magnetic attitude control has been made. The roll motion can be described by a nonlinear differential equation of second order with periodic coefficients. Mitropolskii's method has been employed to compute the asymptotic approximation of the roll equation. The stationary solutions can be represented in terms of the solution of a linear Hill-type equation, which is known and tabulated. It was found that apart from the nonresonant rest position there is a large variety of resonant oscillatory and rotational roll modes. The resonance effect is externally excited by the orbital pitch motion of the spacecraft. Inertial crosscoupling provides the mechanism for exchanging energy between the roll and the transverse modes. The frequency of the roll oscillation as well as the mean rate of the rotational roll motion are found to be certain rational fractions of the orbital frequency. Numerical results are presented in form of amplitude characteristics for the most important resonance modes.

1. Introduction

BECAUSE of its simplicity and reliability the principle of passive magnetic attitude control has been applied to quite a number of national and international European satellite projects in recent years. The principle, first proposed by Fischell[1] makes use of a rigidly mounted permanent magnet to align the reference axis of the spacecraft to the local gradient of the geomagnetic field. The nominal performance of this one-axis control is shown in Fig. 1. In order to damp the librational motions of the satellite, a hysteresis damper is applied, consisting of a number of ferromagnetic rods placed perpendicular to the reference axis.

Previous studies of the attitude dynamics were mainly concerned with the short periodic librations of the system and optimal design of the magnetic control elements with respect to the transient behavior.[1,2] The knowledge about the residual roll motion was rather limited; generally it was assumed that this motion comes to rest after dissipation of the initial kinetic energy.

This paper deals with the analysis of the steady-state roll motion, especially with its long periodic response to the external excitation. The extremely slow nature of these motions allows one to consider the pitch and yaw response as being "loosely coupled" to the roll motion. As a first approximation the forced pitch and yaw motion are introduced as known time-dependent terms. The roll equation appears now as nonlinear differential form of second order, with periodic coefficients.

The analysis of the roll equation was based on the method of asymptotic approximations. Hill's equation was chosen as unperturbed base, since solutions of Hill's equation are available in tabulated form. By a suitable nonlinear transformation the roll equation can be brought into a form convenient for the averaging method developed by Mitropolskii.[3] After introduction of a "slow" timescale and averaging of the standard equations the conditions for the existence of periodic steady-state solutions have been derived. Apart from the purely periodic type of solutions also rotational solutions have been found.

Received June 1, 1970; revision received October 5, 1970. This study was carried out in connection with the European satellite project ESRO-I, which was part of the NASA-ESRO cooperative program of space research.

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In the final section the large variety of periodic and rotational roll modes and their dependency on the parameters of the system is being discussed. The method of approach might also prove useful in analyzing parametric resonance effects occurring with gravity gradient oriented satellites.[4]

2. Equation of the Roll Motion

It is assumed that the satellite can be treated as rigid body. Its dimensions shall be small compared to those of the primary body; hence, the attitude motion can be considered as decoupled from the orbital motion. We furthermore suppose the satellite is moving on a circular and polar orbit. The magnetic field of the Earth shall be approximated by the field of a dipole located at the Earth center; the dipole axis shall coincide with the rotational axis of the Earth.

The attitude of the satellite is described by the relative position of a body fixed-frame to an orbit-fixed frame, Fig. 2. The body-fixed frame (X,Y,Z) is identical to the principle axis system of the body. The Z axis is the reference axis which has to be oriented parallel to the local direction of the geomagnetic field, Fig. 1. The orbit-fixed frame (x,y,z) is defined as follows: the z axis normal to the orbit plane, the y axis parallel to the nodeline of orbital plane and equatorial plane, the x axis completes the right-hand set.

The relative position of the two frames is described by the Eulerian angles (ψ,θ,Φ), which are obtained by the three successive rotations: ψ about the z, θ about x', and Φ around the Z axis. The time history of the Euler angles $\psi(\tau)$, $\theta(\tau)$, $\Phi(\tau)$ is also referred to as pitch, yaw, and roll motion.

In order to derive the equations of motion the Lagrangian formalism is used. The appropriate generalized coordinates of the system are the three Euler angles (ψ,θ,Φ). The angular velocity vector $\boldsymbol{\omega}$ of the rigid body shall be expressed in terms of the generalized coordinates and their time derivatives. In matrix notation one obtains

$$\boldsymbol{\omega} = \begin{bmatrix} \sin\theta\sin\Phi & \cos\Phi & 0 \\ \sin\theta\cos\Phi & -\sin\Phi & 0 \\ \cos\theta & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\Phi} \end{bmatrix} \tag{1}$$

The kinetic energy T of the rigid satellite can now be written as

$$T = \tfrac{1}{2}\boldsymbol{\omega}^T\mathbf{I}\boldsymbol{\omega} \tag{2}$$

where $\boldsymbol{\omega}^T$ denotes the transposed velocity matrix and $\mathbf{I}$ the

Table 1 Coefficients of the roll equation, Eq. (9), for the oscillatory case $\nu = 0$

$g_0 = 4 + 2S_1^2$	$i_n = 0$
$g_1 = 8S_1$	$j_n = 0$
$g_2 = 2S_1^2$	
$R_1 = 0$	
$S_1 = MH_0f_1/(4\Omega^2B + MH_0)$	

diagonalized inertial matrix. The diagonal elements of **I** are the three principal moments of inertia A,B,C associated with the principle axes X,Y,Z.

The potential energy U of the satellite is obtained by integrating the magnetic torque produced by the permanent magnet in the Earth's magnetic field†

$$U = \int_0^\beta (\mathbf{M} \times \mathbf{H})d\beta = MH(1 - \cos\beta)$$

where **M** denotes the magnetic moment vector of the stabilizing magnet pointing in $-Z$ direction. **H** is the local gradient of the geomagnetic field and β the angular distance between **M** and **H**. Using the trigonometric relations indicated in Fig. 2 one can express $\cos\beta$ in terms of the generalized coordinates θ,ψ and the magnetic declination Λ.

Thus one obtains for U

$$U = MH[1 - \sin\theta \sin(\Lambda - \psi)] \tag{3}$$

Amount H and declination Λ of the local field gradient are continuous functions of the time τ, which can be expanded into the series

$$\Lambda = \Lambda_0 + 2\Omega\tau + \sum_{n=1}^{m} f_n \sin 2n\Omega\tau$$
$$H = H_0\left(1 + \sum_{n=1}^{m} h_n \cos 2n\Omega\tau\right) \tag{4}$$

where Ω is the orbital frequency, and f_n,h_n are Fourier coefficients depending on orbital parameters.[5] The energy dissipation of the hysteresis-damper shall be taken into account by introduction of the dissipative function F depending on the roll rate and an equivalent damping factor K

$$F = K\dot{\Phi}^2 \tag{5}$$

Combining Eqs. (1) and (2) and Eqs. (3) and (4) the Lagrangian $L = T - U$ is expressed as function of the generalized coordinates (ψ,θ,Φ) and time. The equations of motion follow from Lagrange's extended equations.[5] The complete system of the equations of motion does not easily lend itself to analytical treatment. But because of the relatively low level of the roll rates considered here, some simplifications are justified.

Table 2 Coefficients of the roll equation, Eq. (9), for the rotational case $\nu = 2$

$g_0 = (S_1 - R_1)^2$	$i_1 = g_1$
$g_1 = 4(S_1 - R_1)$	$i_2 = g_2$
$g_2 = 4 + 2(S_1^2 - R_1^2)$	$i_3 = 4(S_1 + R_1)$
$g_3 = 4(S_1 - R_1)$	$i_4 = g_4$
$g_4 = (S_1 + R_1)^2$	$j_1 = 8R_1$
	$j_2 = 4S_1R_1$
$R_1 = -2MH_0f_1\nu\Omega^2C/D_1$	
$S_1 = MH_0f_1(MH_0 + 4\Omega^2C)/D_1$	
$D_1 = [MH_0 - 2\Omega^2(A + B)](MH_0 + 4\Omega^2C) - 4\nu^2\Omega^4C^2$	

† Terms due to gravitational forces have been omitted in the present study; their influence is of secondary order in cases of strong magnetic control.

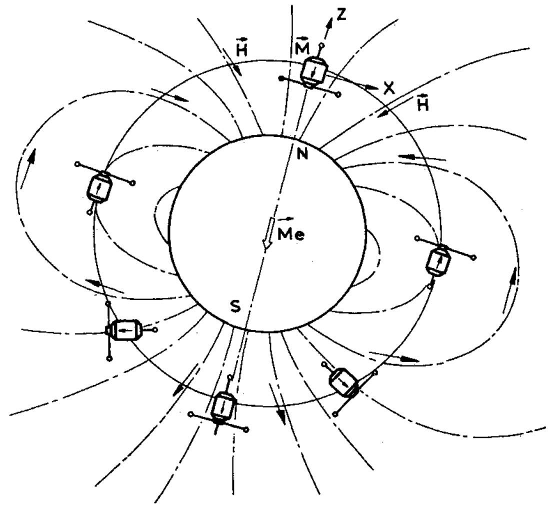

Fig. 1 Nominal attitude of a satellite with passive magnetic stabilization of the Z axis. M = magnetic dipole of satellite, M_e = dipole of the Earth magnetic field.

Numerical tests performed with a computer simulation of the complete dynamic system have shown that at low roll rates ($\sim 10^{-3}$ rad/sec) the pitch motion $\psi(\tau)$ and the yaw motion $\theta(\tau)$ are nearly independent from the roll motion $\Phi(\tau)$. This result is due to the fact that the coupling torques with respect to the ψ and θ motion derived from the kinetic energy Eq. (2) are by one order of magnitude smaller than the magnetic torques derived from Eq. (3).

As a first approximation the pitch and yaw motion might be calculated independent of the roll motion. Because of the hysteresis damping the natural modes of pitch and yaw motion (eigen-frequencies $\sim 10^{-2}$ rad/sec) are damped out soon. What remains are the stationary solutions of the low frequency 2Ω due to the forcing terms Eq. (4) (Ref. 5)

$$\psi(\tau) = 2\Omega\tau + \sum_1^m S_n \sin 2n\Omega\tau$$
$$\theta(\tau) = \frac{\pi}{2} + \sum_1^m R_n \cos 2n\Omega\tau \tag{6}$$

The orbital frequency Ω for the usual near-Earth orbits is in the order of $\sim 10^{-3}$ rad/sec. For the present analysis it is sufficient to retain only the basic amplitudes S_1,R_1, as given in Tables 1 and 2.

Now $\psi(\tau)$ and $\theta(\tau)$ can be introduced as purely time-dependent terms into the Lagrangian $L = T - U$, which remains a function of Φ and τ only. The equation for the roll

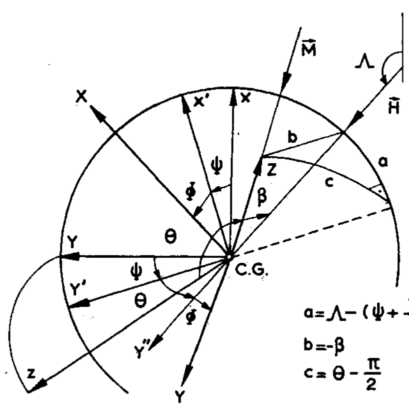

Fig. 2 Definition of Euler angles (ψ,θ,ϕ) and relative position of satellite dipole M to the local field vector H.

motion follows from Lagrange's equation

$$(d/d\tau)(\partial L/\partial\dot{\Phi}) - \partial L/\partial\Phi = -\partial F/\partial\dot{\Phi}$$

together with Eqs. (1–5) as

$$C\ddot{\Phi} + 2K\dot{\Phi} + 0.5(B - A)\{\dot{\psi}^2(\tau)\sin^2\theta(\tau) - \dot{\theta}^2(\tau)\}\sin 2\Phi + (B - A)\dot{\psi}(\tau)\dot{\theta}(\tau)\sin\theta(\tau)\cos 2\Phi - C\dot{\psi}(\tau)\dot{\theta}(\tau)\sin\theta(\tau) + C\ddot{\psi}(\tau)\cos\theta(\tau) = 0 \quad (7)$$

Because of the possible appearance of rotational solutions the roll angle Φ has to be of the general form

$$\Phi(\tau) = \nu\Omega\tau + \phi(\tau) \quad (8)$$

where $\phi(\tau)$ stands for the purely oscillating part; the persistent part of the roll rate is expressed as a rational fraction ν of the orbital frequency Ω [see Eq. (37)]. After introduction of Eqs. (6) and (8) into Eq. (7) and normalizing time and amplitude by $t = \Omega\tau$, $x = 2\phi$ we obtain the normalized roll equation

$$\ddot{x} + 2\bar{K}\dot{x} + \Delta G(2t)\sin x = \Delta I(2t)\cos x + J(2t) \quad (9)$$

where $\bar{K} = K/C\Omega$, $\Delta = (B - A)/C$, and G,I,J stand for the finite Fourier series of the real period π

$$G(2t) = \sum_{n=0}^{m} g_n \cos 2nt, \qquad I(2t) = \sum_{n=1}^{m} i_n \sin 2nt,$$

$$J(2t) = j_1 \sin 2t + j_2 \sin 4t$$

The coefficients g_n,i_n,j_n are functions of the pitch and yaw amplitudes S_1,R_1 and of the roll coefficient ν. Two typical sets of coefficients for the oscillatory case ($\nu = 0$) and for the rotational case ($\nu = 2$) are shown in Tables 1 and 2.

3. Equation of the Asymptotic Approximation

In the following sections we shall investigate the steady-state solutions of the roll equation applying the method of asymptotic approximations.[3] Previous solutions of equations of type shown in Eq. (9) imposed rather stringent conditions on the periodic function $G(2t)$. Poincare[6] developed a perturbation scheme based on the unperturbed form (Mathieu's equation)

$$\ddot{y} + (a + b\cos 2t)y = 0$$

This method, however, requires stability of the generating solution, i.e., the parameter-pair (a,b) has to belong to a stable region of Mathieu's stability chart.[7]

The same restriction holds for the solution presented by Mitropolskii and Lykova[3] who used the method of asymptotic approximation. But Skalak and Yarymovych[8] showed experimentally that under presence of damping subharmonic oscillations will be sustained which require the parameters (a,b) to be located in the unstable regions.

In the present study a method of solution is developed without restricting the coefficients g_n of the periodic function $G(2t)$ of the roll equation, Eq. (9). The constraints that have to be retained concern the damping and the forcing term on the right-hand side of Eq. (9), which both are supposed to be small of order ϵ. The oscillations of the roll angle around its stationary value shall be small enough to allow for an expansion of the term $\sin x$ in powers of x, regarding the nonlinear terms as perturbing force.

With these assumptions the unperturbed form of the roll equation, Eq. (9), can be recognized as Hill's equation. Solutions of Hill's equation are known; they can be presented as infinite series of trigonometric or hyperbolic functions. The coefficients of these expansions have been tabulated by several authors.[7]

For the present investigation the solution suggested by Whittaker is applied, which has the general form

$$y = z_1 e^{\mu t} W_1(t,\sigma) + z_2 e^{-\mu t} W_2(t,\sigma) \quad (10)$$

z_1,z_2 are integration constants, μ is the characteristic exponent, σ the characteristic phase. It follows from Floquets theorem that (μ,σ) are imaginary numbers for stable solutions of Hill's equation, for unstable solutions they have to be real. W_1,W_2 are periodic functions of time with period π or 2π expandable in circular functions

$$W_1 = \Sigma_m s_m \sin(mt - \sigma) + c_m \cos(mt - \sigma) \quad (11a)$$

$$W_2 = \Sigma_m s_m \sin(mt + \sigma) + c_m \cos(mt + \sigma) \quad (11b)$$

The sequence for m depends on the number p of the stability region INp of Hill's equation:

$m = 0,2,4, \ldots$ even integers for p even

$m = 1,3,5, \ldots$ odd integers for p odd

also $s_m = 1$, $c_m = 0$ for $m = p$.

The exponent μ and the coefficients s_m,c_m are given as power series of g_n, where g_n are the Fourier coefficients of the periodic function $G(2t)$, Eq. (9). For details we refer the reader to the quite extensive monograph on the theory of Hill's equations by MacLachlan.[7]

In view of the algorithm developed in Sec. 4 we aim at a two-parametric form of Hill's equation. Taking into account the assumptions previously made, we rewrite Eq. (9) in a form appropriate for the perturbation approach

$$\ddot{x} + 2\epsilon k\dot{x} + [a + bH(2t)][x + \epsilon f(x)] = \epsilon F(2t,x) \quad (12)$$

where ϵ is a parameter of sufficient smallness, and

$$\epsilon k = \bar{K}, \quad a = \Delta g_0, \quad b = \Delta, \quad \epsilon f(x) = -(\tfrac{1}{6})x^3$$

$$\epsilon F(2t,x) = \Delta I(2t)\cos x + J(2t), \quad H(2t) = G(2t) - g_0$$

assuming that $x - (\frac{1}{6})x^3$ is an acceptable first approximation for $\sin x$. To account for the influence of the perturbations, we introduce the "detuning" a_1,b_1 of the unperturbed parameters a_0,b_0

$$a = a_0 + \epsilon a_1, \quad b = b_0 + \epsilon b_1 \quad (13)$$

The damping term $2\epsilon kx$ shall be removed by the transformation

$$x = ye^{-\epsilon kt} \quad (14)$$

Introduction of Eqs. (13) and (14) into Eq. (12) leads to

$$\ddot{y} + [a_0 + b_0 H(2t)]y = \epsilon Y(y,t) \quad (15)$$

where

$$Y = \{-[a_1 + b_1 H(2t)]x - [a_0 + b_0 H(2t)]f(x) + F(2t,x)\}e^{\epsilon kt}$$

(the term ϵ^2k^2y of Y has been dropped because of its smallness of second order). The unperturbed form of Eq. (15) reduces for $\epsilon = 0$ to Hill's equation

$$\ddot{y} + [a_0 + b_0 H(2t)]y = 0 \quad (16)$$

The roll equation has taken now the form of Eq. (15), convenient for the application of Mitropolskii's method of asymptotic approximation.[3] For $\epsilon = 0$ the solution of Eq. (15) is given by Eq. (10). Because of the influence of the perturbing term (ϵY) the quantities z_1,z_2 become time dependent themselves. To determine the unknown functions $z_1(t),z_2(t)$ Eq. (15) is transformed into a system of differential equations for z_1,z_2 called the "standard form." The suitable transformation is

$$y = z_1 e^{\mu t} W_1 + z_2 e^{-\mu t} W_2 \quad (17a)$$

$$dy/dt = z_1 e^{\mu t}(\mu W_1 + \dot{W}_1) + z_2 e^{-\mu t}(-\mu W_2 + \dot{W}_2) \quad (17b)$$

where W_1, W_2 are the periodic functions of Whittaker's solution, Eq. (10); the dots denote derivatives with respect to time. Applying Eq. (17) to Eq. (15) and taking advantage of the fact that $e^{\mu t}W_1$ and $e^{-\mu t}W_2$ are particular solutions of the unperturbed equation, Eq. (16), one obtains the standard form

$$dz_1/dt = \epsilon(1/D)Y(z_1,z_2,t)W_2(t)e^{-\mu t} \quad (18a)$$

$$dz_2/dt = -\epsilon(1/D)Y(z_1,z_2,t)W_1(t)e^{\mu t} \quad (18b)$$

where Y is the perturbing function of Eq. (15) after substitution of y according to Eq. (17). D represents the Wronskian of the unperturbed equation, Eq. (16),

$$D = W_1\dot{W}_2 - W_2\dot{W}_1 + 2\mu W_1 W_2$$

(D is constant for any time and can thus be determined at the initial time t_0).

The essential idea behind the transformation into standard form is the separation of slowly and rapidly varying terms. z_1, z_2 are slowly varying since their time derivative is proportional to the small quantity ϵ. We introduce the slow time t_s which is related to the "fast time" t by $t_s = \epsilon t$.

The steady-state conditions derived later require a value μ that is in the order of the damping factor ϵk. We introduce the slow time into the exponential functions of Eq. (18)

$$e^{\mu t} = e^{\lambda t_s} \quad \text{with} \quad \mu = \epsilon\lambda \quad (19)$$

where λ is a quantity close to k.

According to Mitropolskii,[3] the first approximations z_{10}, z_{20} are obtained by averaging Eqs. (18). Let M_t be the operator for averaging a function F over the fast time t,

$$M_t\{F(t,t_s)\} = \lim_{t\to\infty} (1/T) \int_0^T F(t,t_s)dt \qquad (t_s = \text{const})$$

then the average Eqs. (18) become

$$dz_{10}/dt_s = M_t\{(1/D)YW_2e^{-\lambda t_s}\} \quad (20a)$$

$$dz_{20}/dt_s = M_t\{-(1/D)YW_1e^{\lambda t_s}\} \quad (20b)$$

where Y is obtained from Eq. (15), after substitution of x according to Eq. (14), and y according to Eq. (17) as

$$Y = -Q_1(z_{10}e^{\lambda t_s}W_1 + z_{10}e^{-\lambda t_s}W_2) + F(2t,x)e^{kt_s} + Q_0(z_{10}e^{\lambda t_s}W_1 + z_{20}e^{-\lambda t_s}W_2)^3e^{-2kt_s} \quad (21)$$

with

$$Q_0 = a_0 + b_0H(2t), \quad Q_1 = a_1 + b_1H(2t) \quad (22)$$

4. Determination of the Periodic Solutions

In the following section the algorithm for computation of the periodic steady-state solutions shall be derived. As a first steady-state condition periodicity of $x(t)$ is requested. Since W_1, W_2 are periodic in t, it follows from Eqs. (14) and (17)

$$\mu = \epsilon k \quad \text{for} \quad \mu > 0, \quad \mu \text{ real}$$

or expressed as slow-time coefficient with Eq. (19):

$$\lambda = k \quad (23)$$

The second steady-state condition can be found from the standard equations, Eqs. (20). For the oscillatory case, the time dependent functions $I(2\tau)$ and $J(2\tau)$ of the roll equation, Eq. (9), vanish (see Table 1); hence the term $F(2t,x)$ of Eq. (21) is identical zero.

The standard equations, Eqs. (20), take with Eqs. (21), and (23) the form

$$D(dz_{10}/dt_s) = M_t[\{z_{10}^3Q_0W_1^3W_2 - z_{10}Q_1W_1W_2\} + \{3z_{10}^2z_{20}Q_0W_1^2W_2^2 - z_{20}Q_1W_2^2\}e^{-2kt_s} + \{3z_{10}z_{20}^2Q_0W_1W_2^3\}e^{-4kt_s} + \{z_{20}^3Q_0W_2^4\}e^{-6kt_s}] \quad (24a)$$

$$D(dz_{20}/dt_s) = M_t[\{-z_{10}^3Q_0W_1^4 + z_{10}Q_1W_1^2\}e^{2kt_s} + \{-3z_{10}^2z_{20}Q_0W_1^3W_2 + z_{20}Q_1W_1W_2\} - \{3z_{10}z_{20}^2Q_0W_1^2W_2^2\}e^{-2kt_s} - \{z_{20}^3Q_0W_1W_2^3\}e^{-4kt_s}] \quad (24b)$$

The second steady-state condition to be imposed on Eqs. (24) is

$$dz_{10}/dt = 0, \quad dz_{20}/dt = 0, \quad \text{for } t_s \to \infty$$

leading to a system of algebraic equations for the determination of the steady-state solution. Provided k is below a certain limit, the following conditions are found by inspection of Eq. (24)

$$\begin{aligned} M_t\{z_{10}^3Q_0W_1^3W_2 - z_{10}Q_1W_1W_2\} &= 0 \\ M_t\{z_{10}^3Q_0W_1^4 - z_{10}Q_1W_1^2\} &= 0, \quad z_{20} = 0 \end{aligned} \quad (25)$$

The validity of Eqs. (25) has been checked by solving the system, Eqs. (24), analytically for large τ, and by programing Eqs. (24) for an analog computer.[5] The limit value for k was found as

$$k < (1/D)M_t\{Q_1W_1W_2\} \quad (26)$$

Before averaging Eqs. (25), it is convenient to define the following Fourier series for the products of W_1, W_2:

$$W_1^2 = \Sigma_m(A_m \cos mt + A_{m+1} \sin mt) \quad (27a)$$

$$W_1W_2 = \Sigma_m(B_m \cos mt + B_{m+1} \sin mt) \quad (27b)$$

$$W_1^4 = \Sigma_m(E_m \cos mt + E_{m+1} \sin mt) \quad (27c)$$

$$W_1^3W_2 = \Sigma_m(F_m \cos mt + F_{m+1} \sin mt) \quad (27d)$$

$$(m = 0,2,4,\ldots)$$

A_m to F_m are functions of s_m, c_m of Eq. (11), they have been tabulated[5] up to $m = 6$. Introduction of Eqs. (22) and (27) into Eqs. (25) and computing the average M_t leads to a system of linear equations for a_1 and b_1. Written in matrix notation one obtains the symmetric form

$$\mathbf{P}\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = Z_{10}^2\mathbf{Q}\begin{bmatrix} a_0 \\ b_0 \end{bmatrix} \quad (28)$$

where $\mathbf{P}$ and $\mathbf{Q}$ are square-matrices composed of the coefficients g_n from $G(2t)$ and A_m to F_m of Eqs. (27). For the oscillatory case we obtain simply

$$\mathbf{P} = \begin{bmatrix} 2A_0 & (g_1A_2 + g_2A_0) \\ 2B_0 & (g_1B_2 + g_2B_0) \end{bmatrix} \quad \mathbf{Q} = \begin{bmatrix} 2E_0 & (g_1E_2 + g_2E_4) \\ 2F_0 & (g_1F_2 + g_2F_4) \end{bmatrix}$$

the coefficients g_1, g_2 are given in Table 1.

The conditions for Z_{20} and k become

$$Z_{20} = 0, \quad k < (1/2D)\{2B_0a_1 + (g_1B_2 + g_2B_4)b_1\} \quad (29)$$

Z_{10}, Z_{20} denote the steady-state values for large t_s. Equation (28) describes the relation between detuning (a_1, b_1) and amplitudes Z_{10} of the nonlinear oscillation. By solving Eq. (28) for (a_1, b_1) and taking into account the definition of the detuning Eq. (13) we obtain the relation between the linear and nonlinear parameters

$$\begin{bmatrix} a \\ b \end{bmatrix} = (\mathbf{E} + \epsilon Z_{10}^2\mathbf{P}^{-1}\mathbf{Q})\begin{bmatrix} a_0 \\ b_0 \end{bmatrix} \quad (30)$$

where $\mathbf{P}^{-1}$ denotes the inverse of the matrix $\mathbf{P}$ and $\mathbf{E}$ the unit matrix.

The parameters (a_0, b_0) of the unperturbed equation are determined by the condition to constitute a "Whittaker solution" as defined in Eq. (10) and to fulfill the steady state condition of Eq. (23). Thus, Eqs. (30) enable us to express the amplitude Z_{10} as function of the nonlinear parameters (a,b) only. The problem posed initially requires to represent Z_{10} as a function of the crosscoupling Δ of Eq. (9). The relation

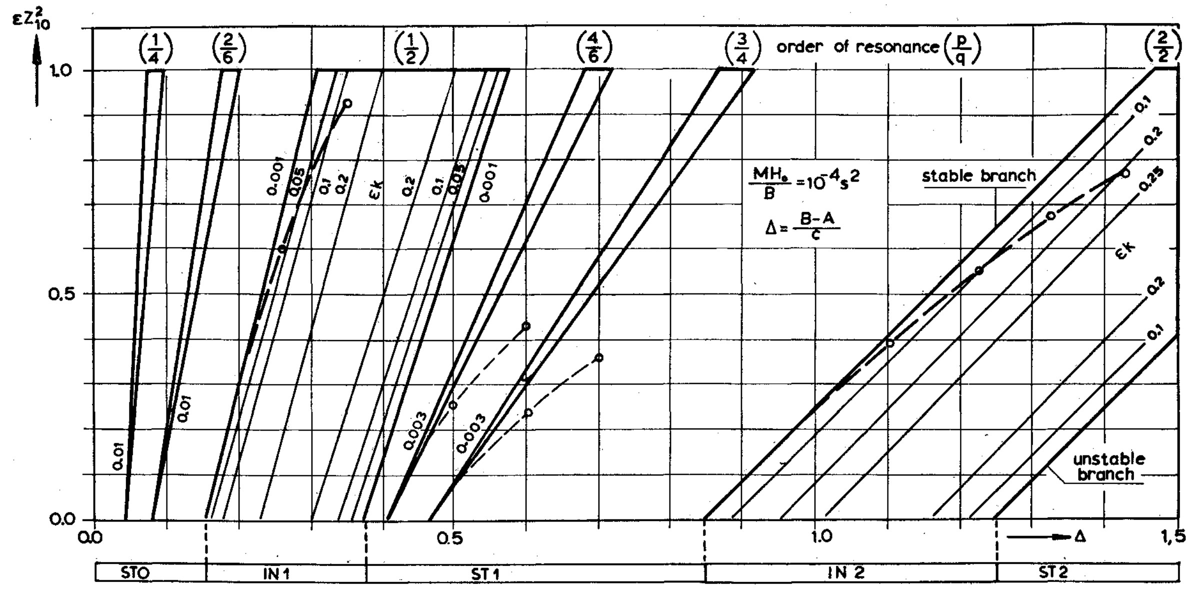

Fig. 3 Amplitude characteristic for oscillatory roll modes. Δ = **crosscoupling factor,** ϵk = **damping factor.**

between Δ and (a,b) was defined in (12) as

$$a = \Delta g_0 \quad \text{and} \quad b = \Delta$$

Applying this to Eq. (30) and solving for $(\Delta, \epsilon Z_{10}^2)$ yields

$$\begin{bmatrix} \Delta \\ \epsilon Z_{10}^2 \end{bmatrix} = \mathbf{R}^{-1} \begin{bmatrix} a_0 \\ b_0 \end{bmatrix} \tag{31}$$

where $\mathbf{R}$ denotes the 2×2 matrix $\mathbf{R} = [\mathbf{R}_1, \mathbf{R}_2]$ which consists of the two column-blocks

$$\mathbf{R}_1 = \begin{bmatrix} g_0 \\ 1 \end{bmatrix} \quad \mathbf{R}_2 = -\mathbf{P}^{-1}\mathbf{Q} \begin{bmatrix} a_0 \\ b_0 \end{bmatrix}$$

For computational reasons it is of advantage to keep the parametric representation of Eq. (31).

The application of the algorithm shall be demonstrated for the case of subharmonic oscillations of order $(p/q) = (\frac{1}{2})$. For a given positive damping $\epsilon k = \bar{K}$ the characteristic exponent is fixed by Eq. (23) as $\mu = \epsilon k$. Since μ is real and the periodic solution shall be of order $(\frac{1}{2})$ (basic period 2π) the Whittaker solution used for the asymptotic approximation has to be taken for the first unstable region $IN1$ of Hill's equation. The relation between the linear parameters (a_0, b_0) on one side and (μ, σ) on the other is given in form of two power series in b_0

$$\mu = \sum_{n=1}^{\infty} f_{1n}(\sigma, g_m) b_0^n \tag{32}$$

$$a_0 = \sum_{n=1}^{\infty} f_{2n}(\sigma, g_m) b_0^n \tag{33}$$

The coefficients f_{1n}, f_{2n} have been tabulated by Hayashi.[9] In Table 3 the first three coefficients for the region $IN1$ are shown as an example. The numerical computation starts by selecting an initial value for the independent parameter b_0. From Eq. (32) the phase σ can be computed substituting $\mu = \epsilon k = \bar{K}$. a_0 follows from Eq. (33). The coefficients s_m, c_m of W_1 in Eq. (11) are similar power series in b_0 (Ref. 9).

$$s_m = \sum_{n=1}^{\infty} f_{3n}(\sigma, g_m) b_0^n, \quad c_m = \sum_{n=1}^{\infty} f_{4n}(\sigma, g_m) b_0^n$$

Now the coefficient-matrices $\mathbf{P}$, $\mathbf{Q}$ of Eq. (28) can be calculated; and finally from Eq. (31) one obtains a set of $(\Delta, \epsilon Z_{10}^2)$ values.

To each positive value of Δ belong two real amplitudes Z_{10} corresponding to the stable respectively unstable periodic solution (see Fig. 3). The final steady-state solution $\bar{x}$ is obtained after substitution of Eq. (17) into Eq. (14) and ob-

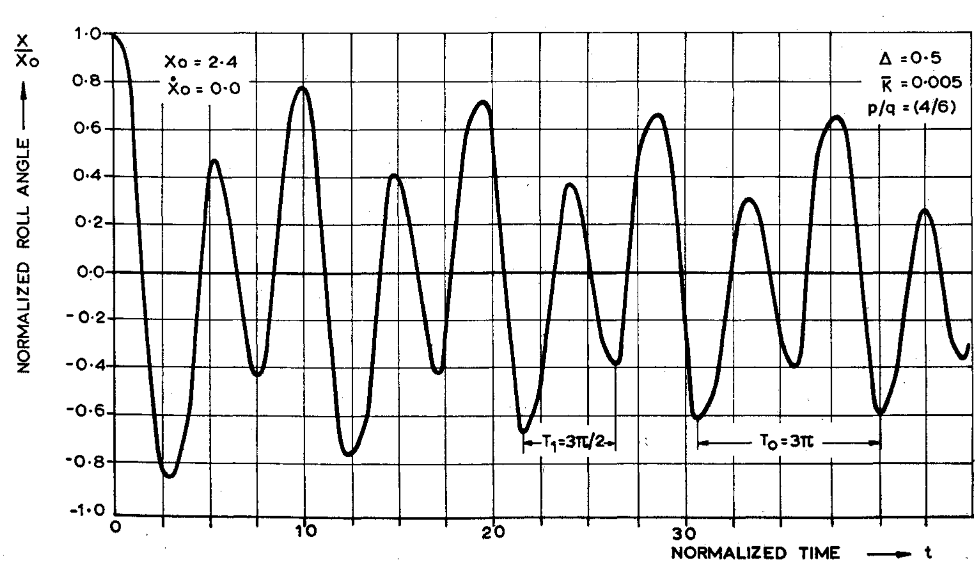

Fig. 4 Acquisition of the stable resonance mode ($\frac{4}{6}$) (numerical integration). The solution is of even order with normalized frequencies $\omega_m = (m/q)2$, $m = 0, 2, 4, \ldots \infty$, and dominating frequency $\omega_1 = (\frac{4}{6})2$.

Table 3 Coefficients of the expansions $\mu(b_0)$ and $a_0(b_0)$ for Hill's equation in the first unstable region $IN1$ (Ref. 9)

$$f_{11} = \tfrac{1}{4}\sin 2\sigma$$
$$f_{12} = \tfrac{1}{4}(\tfrac{1}{8}g_2 + \tfrac{1}{24}g_2g_3)\sin 2\sigma$$
$$f_{13} = \tfrac{1}{8}(\tfrac{-3}{128} + \tfrac{5}{384}g_3 + \tfrac{7}{288}g_2^2 + \tfrac{17}{4608}g_3^2)\sin 2\sigma + \tfrac{1}{1024}g_2\sin 4\sigma$$

$$f_{21} = \tfrac{1}{2}\cos 2\sigma$$
$$f_{22} = \tfrac{1}{4}[-\tfrac{1}{4} - \tfrac{1}{6}g_2^2 - \tfrac{1}{16}g_3^2 + (\tfrac{1}{4}g_2 + \tfrac{1}{12}g_2g_3)\cos 2\sigma + \tfrac{1}{8}\cos 4\sigma]$$
$$f_{23} = \tfrac{1}{8}(-\tfrac{1}{64}\cos 2\sigma + \tfrac{5}{64}g_2\cos 4\sigma)$$

serving conditions Eqs. (29) as

$$\bar{x} = (Z_{10}W_1)_{IN1}$$

Introducing W_1 associated with Hill's region $IN1$, we find

$$\bar{x} = Z_{10}[\cos(t-\sigma) + \Sigma_m s_m \sin(mt-\sigma) + \Sigma_m c_m \cos(mt-\sigma)], \quad m = 3,5,7, \ldots$$

which represents a subharmonic solution of the roll equation, Eq. (9), with basic period 2π.

The general periodic solution for the resonance mode of order (p/q) having the dominant frequency $f = 2(p/q)$ and a basic period of $T_0 = q\pi$ (p odd, q even), respectively, $T_0 = q\pi/2$ (p even, q even) can be represented as

$$\bar{x}(t) = (Z_{10}\Sigma_m[s_m \sin\{(m/q)2t - \sigma\} + c_m \cos\{(m/q)2t - \sigma\}])_{INp} \quad (34)$$

$$m = 0,2,4, \ldots \text{ even, for } p \text{ even, } q \text{ even}$$

$$m = 1,3,5, \ldots \text{ odd, for } p \text{ odd, } q \text{ even}$$

$$\text{and } s_m = 1,\ c_m = 0 \text{ for } m = p$$

The index INp shall indicate that the expansions for μ, a_0, s_m, c_m have to be taken for the region INp of Hill's equation. The value of Z_{10} is found by application of the algorithm in the aforementioned way. The relation between true and normalized roll angle was defined in Eq. (9) as

$$\phi = \bar{x}/2$$

The rotational solutions of the roll equation Eq. (7) are of the general form defined in Eq. (8)

$$\Phi = \nu\Omega\tau + \phi(\tau) \quad (35)$$

The resonant rotational solution is established when the system returns to identical energetic state after completing integer numbers of half-turns. Thus the following boundary conditions characterize this case:

$$\Phi(qT) = \Phi(0) + n\pi \quad (36a)$$

$$\dot{\Phi}(qT) = \dot{\Phi}(0) \qquad n = \text{integer} \quad (36b)$$

$T = \pi/\Omega$ denotes the basic period of the driving terms $\psi(\tau)$ and $\theta(\tau)$ given in Eq. (6). The even integer q characterizes the basic period of the superimposed oscillation. The coefficient ν is found from Eqs. (35) by applying Eqs. (36)

$$\nu = (n/q) \qquad n = \text{integer} \quad (37)$$

The determination of the superimposed periodic oscillation follows exactly the same scheme as outlined for the oscillatory case. Once a particular value for ν has been chosen the coefficients of $I(2t)$ and $J(2t)$ in Eq. (9) can be computed. An example for $\nu = 2$ is given in Table 2; these terms appear as periodic driving force $F(2t,x)$ in Eq. (12) their normalized basic period is π. They will be of influence on those periodic solutions for which the averaging

$$M_t\{W_1(t)F(2t,x)\}; \quad M_t\{W_2(t)F(2t,x)\}$$

results in a nonzero value and contribute thus to the steady-state solutions of the standard form Eq. (24).

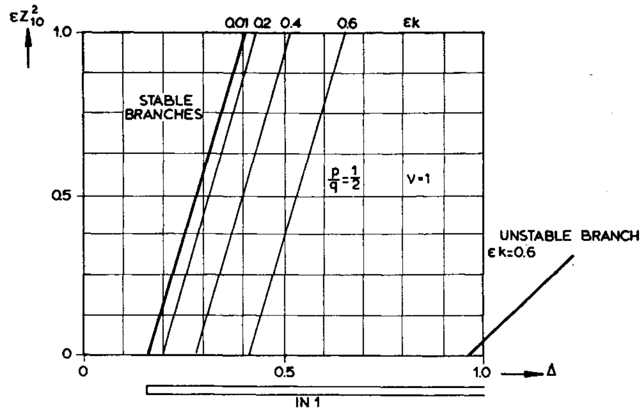

Fig. 5 Amplitude characteristic for the rotational roll mode $\nu = 1$ (one revolution per orbit).

5. Results of Computations

The algorithm developed in the previous section has been used to calculate the amplitude characteristics for a typical satellite of $(MH_0/B) = 10^{-4}$ sec^2 in a 500 km circular orbit with $H_0 = 0.345$ oe, $f_1 = 0.334$, $h_1 = 0.324$. Most effort has been spent on the computation of the characteristics $\epsilon Z_{10}^2(\Delta)$ for oscillatory roll motions ($\nu = 0$) as plotted in Fig. 3, to show the manifold of possible resonance models.

In order to check the stability and the accuracy of the approximate solutions, the roll equation, Eq. (9), has been integrated numerically in the vicinity of the periodic trajectory, (see dashed curve of Fig. 3). It was found that the "upper" branch of the amplitude characteristic of order (p/q) belongs to stable solutions, the "lower" branch to unstable solutions. The accuracy of the first approximation is satisfactory for the resonance cases of order $(p/2)$ up to $\epsilon Z_{10}^2 = 0.8$. For fractional order resonance $(p/4)$ and $(p/6)$ the accuracy is one order lower.

Looking at the entirety of periodic solutions of Eq. (9), there are some interesting general properties that shall be mentioned before we pass on to the physical interpretation.

Let us consider the solutions $\bar{x}$ given by Eq. (34) as elements of the function space $U_q[0,q\pi]$ defined in the interval $[0,q\pi]$. All solutions of order (p/q) with common index q shall be comprised by the set $U_q[0,q\pi]$. Thus $U_q[0,q\pi]$ contains all even and odd solutions of the common basic period $T_0 = q\pi$ (for p odd integer), respectively $T_0 = q\pi/2$ (for p even). Further on the subset $S_{p,q} \subset U_q$ shall contain all those solutions $\bar{x} \in S_{p,q}$ which are of the particular resonance order (p/q).

Using these definitions one finds one distinguished set $U_2[0,2\pi]$ for the lowest possible value of q, $(q = 2)$. All solutions $\bar{x} \in U_2$ approach solutions of linear parametric reso-

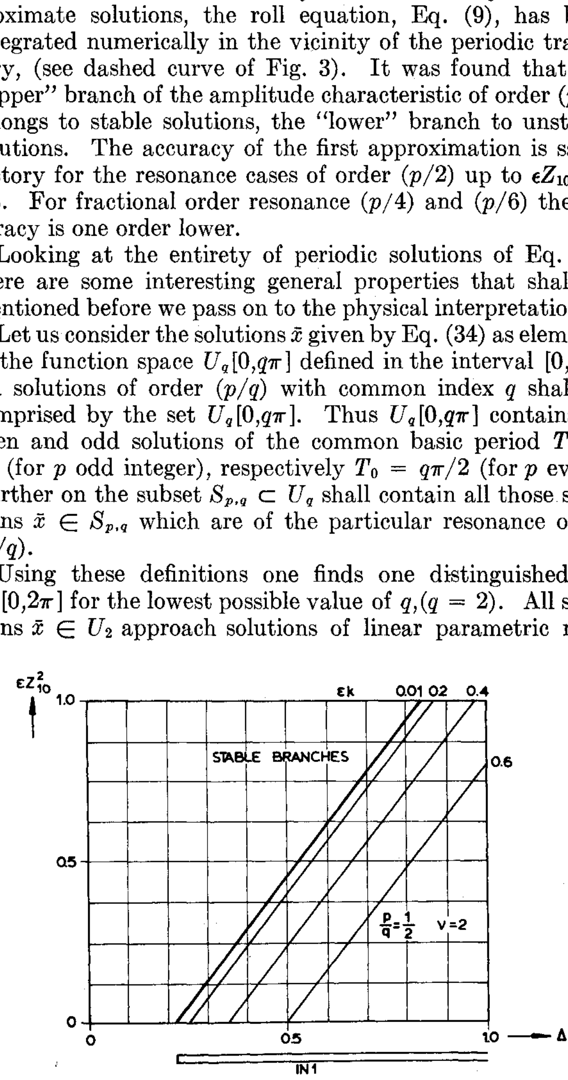

Fig. 6 Amplitude characteristic for the rotational roll mode $\nu = 2$ (two revolutions per orbit).

nance for extremely small amplitudes ($\epsilon \to 0$). The amplitude curves associated with the subsets $S_{p,2} \subset U_2$ originate in the regions INp of the Δ scale of Fig. 3. Therefore U_2 is called the "parametric" set of solutions. With increasing damping ϵk the stable and unstable branch of the amplitude curve approach each other; there is a critical damping above which no periodic solution exists [see Eq. (26)].

All other sets $U_{qn}[0,q_n\pi]$, where $q_n > 2$, contain solutions which are due to purely nonlinear resonance. For vanishing nonlinearity ($\epsilon \to 0$) these solutions cannot exist anymore. The amplitude curves of the solutions $\bar{x} \in S_{p,qn}$ are located between amplitude curves of $\bar{x} \in S_{p,2}$ in the representation of Fig. 3. There exists a "cut-off" amplitude depending on ϵk below which no periodic solution can exist.

6. Conclusions

According to the principle of passive magnetic control the satellite is forced into a nonuniform pitch motion. The basic frequency of this motion is twice the orbital frequency. If the spacecraft is of inertial unsymmetry ($A \neq B$, $\Delta \neq 0$) an energy exchange between roll motion and pitch motion is provided through inertial coupling. For particular values of the crosscoupling factor

$$\Delta = (B - A)/C$$

external excitation of resonant roll motions is possible. Generally three kinds of stationary roll solutions have been found: 1) equilibrium or rest position; 2) resonant oscillatory motion; and 3) resonant rotational motion.

In the following, the three typical roll modes and their dependency on the crosscoupling, Δ shall be discussed. Note that Δ is physically realistic only for $\Delta \leqq 1$. Nonzero damping $\epsilon k > 0$ is assumed.

1) The equilibrium position of the roll angle is of limited asymptotic stability if Δ is located inside a ST region of the Δ scale of Fig. 3; for Δ inside a IN region the rest position is unstable. That means the usual hysteresis damper ($\epsilon k \ll 1$) can only provide a stable rest position of the roll angle if the crosscoupling factor Δ has been adjusted to a value inside ST. The equilibrium is defined by $\Phi = 0 \pm n\pi$; in this position the axis of maximal moment of inertia (Y axis) is directed normal to the orbit plane.

2) The oscillatory mode has to be subdivided into two groups of different stability properties. Periodic roll oscillations of strong stability with respect to the perturbing vector $(\Phi_0,\dot{\Phi}_0)$ are obtained if the solutions belong to the "parametric" set $U_2[0,2\pi]$. The crosscoupling Δ has thus to be located inside regions $IN1$ or $IN2$ of Fig. 3. For Δ inside $IN1$ the stable roll oscillations (upper branch) are of subharmonic order ($\frac{1}{2}$), i.e., their basic period is equal to the orbital period $T = 2\pi/\Omega$. For Δ inside $IN2$ the roll oscillation is periodic with half-orbital period $T = \pi/\Omega$. Oscillatory roll modes of low stability are obtained if the solution belongs to any set $U_{qn}[0,q_n\pi]$. For small Δ values inside $ST0$ subharmonic oscillations of two and three times orbital period have been found, whereas Δ inside $ST1$ can lead to fractional order solutions. This type of roll modes can only be initiated if the initial vector $(\Phi_0,\dot{\Phi}_0)$ is very close to the steady-state trajectory defined by the solution Eq. (34). An example for the transition to the stable roll mode of the fractional order ($\frac{4}{6}$) is shown in Fig. 4. The solution was obtained by numerical integration of the roll equation, Eq. (9).

3) Rotational roll motions can be initiated if the initial rate comes close to the mean roll rate $\langle\dot{\Phi}\rangle = \nu\Omega$, where $\nu = n/q$ as defined in Eq. (37). Reasonably stable solutions were found for integer values of ν and $q = 2$; that means the spacecraft will perform a continuous motion of ν revolutions around its roll axis superimposed by an oscillation of the period $T = 2\pi/\Omega$. One condition for the existence of the rotational solution is the stability of the superimposed oscillation. These conditions can be retrieved from the amplitude curves; two examples for $\nu = 1$ and 2 are shown in Figs. 5 and 6. Provided the initial perturbation is of sufficient level, these rotational modes can only be sustained if Δ has been adjusted to values inside the region $IN1$ of Figs. 5 and 6, respectively.

The present analysis has been performed assuming a circular and polar orbit. Relaxing these conditions leads to changes of the forcing terms $\Lambda(\tau)$, $H(\tau)$,Eq. (4). With polar orbits the exciting pitch motion has its maximum strength, but is reduced to nearly zero-level for the equatorial orbit. Therefore the described external roll resonance will occur only with near polar orbits which are normally chosen for the magnetically oriented satellite. Deviations from the circularity of the orbit will introduce uneven terms of frequency $(2n - 1)\Omega$ into the expansions Eq. (4). Thus the basic frequency of the forcing terms changes from 2Ω to Ω (or from 2 to 1 in normalized units). The result is an extension of the possible roll modes given by Eq. (34) to solutions of the normalized frequency $\omega_m = (m/q)1$.

In closing we remark that all roll modes found during this analysis are of limited stability. Depending on the parameters of the system there exists a limiting surface around the steady-state trajectory enclosing a finite volume of stable trajectories. If the perturbing state-vector penetrates this surface, the spacecraft changes to another type of resonance mode. But, as current studies show, it seems feasible to preserve a continuous roll motion for a given satellite once the level of environmental perturbations is known.

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